ESPRIT(LS)

Hello

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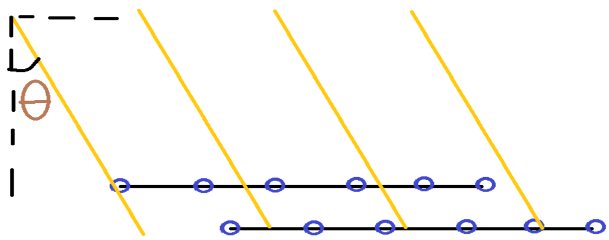
1 Problem Formulation

There are D signal sources, 2 subarrays with elements.
Note:D<m.
The signal is far away from the arrays.
Meanwhile,the 2 subarrays are same totally except their location.
Then, we can go!



2 Deducing

$$\mathbf{X} = \mathbf{AS} + \mathbf{N_x} \tag{1}$$

$\mathbf{X}$ is the data on each element of the first subarray.It is a m $\cdot$ n matrix.m is the number of the elements in each subarray,while n is the number of snapshots.
$\mathbf{A}$ is the array maifold of the first subarray.
N_x is the noise on the first subarray.
Similarily.

$$\mathbf{Y} = \mathbf{A}\boldsymbol{\phi}\mathbf{S} + \mathbf{N_y} \tag{2}$$

$\mathbf{Y}$ is the data on each element of the second subarray.
$\mathbf{A}\phi$ is the arraymainifold of the second subarray.Note:the second arraymanifold is just a rotation of the first arraymanifold.So,ϕ is the rotation operator.
N_y is the noise on the second subarraymanifold.
Looking at the ϕ in more details:

$$\phi = diag\{e^{j\gamma_1}\ e^{j\gamma_2}\ e^{j\gamma_3}\ \ldots\ldots\ e^{j\gamma_D}\} \tag{3}$$

In this equation,$\gamma_k = \omega_i \Delta \sin\theta_k / c$,and ω_i is the central frequency of the $i-th$ signal source.θ_k is the angle of incidence of the i-th source.
Let's simplify it.Assuming

$$\bar{\mathbf{A}} = \begin{bmatrix} \mathbf{A} \\ \mathbf{A}\phi \end{bmatrix}, \mathbf{N_z} = \begin{bmatrix} \mathbf{N_x} \\ \mathbf{N_y} \end{bmatrix}, \mathbf{Z} = \begin{bmatrix} \mathbf{X} \\ \mathbf{Y} \end{bmatrix} \tag{4}$$

Thus, we can denote this:

$$\mathbf{Z} = \bar{\mathbf{A}}\mathbf{S} + \mathbf{N_z} \tag{5}$$

Then,to find the correlation matrix.

$$\mathbf{R}_{zz} = \bar{\mathbf{A}}\mathbf{R}_{ss}\bar{\mathbf{A}}^H + \sigma^2\mathbf{I} \tag{6}$$

$\mathbf{R}_{zz}$ is the correaltion matrix of $\mathbf{Z}$.
$\mathbf{R}_{ss}$ is the correaltion matrix of $\mathbf{S}$.
σ^2 is the square error of the noise.(Here we assume that all the noise are guassian)
$\mathbf{I}$ is the identity matrix.
With all the given information, next step is to find the feature vectors of

$\mathbf{R}_{zz}$.And the linear subspace of the signal is available.Find the D biggest feature numbers and their feature vectors accordingly.
$\mathbf{U}_s$= $[\mathbf{e}_1, \mathbf{e}_2, ..., \mathbf{e}_D]$
Since $\mathbf{U}_s$ is equivlent to the linear subspace expanded by $\bar{\mathbf{A}}$,There must be full-order matrix $\mathbf{T}$ satisfying: $\mathbf{U}_s = \bar{\mathbf{A}}\mathbf{T}$
which means:

$$\mathbf{U}_s = \begin{bmatrix} \mathbf{U}_x \\ \mathbf{U}_y \end{bmatrix} = \begin{bmatrix} \mathbf{AT} \\ \mathbf{A}\boldsymbol{\phi}\mathbf{T} \end{bmatrix} \tag{7}$$

Note:$\mathbf{A}$ is a m×D matrix.$\bar{A}$ is a 2m×D matrix.R$\{E_x\}$,R$\{E_y\}$ and R$\{A\}$ are the same.
Then we can denote $\mathbf{U}_{xy} = [\mathbf{U}_x, \mathbf{U}_y]$
It's a d·2d matrix with d order.Then, there must be a 2d·d matrix F=$[F_x, F_y]$ satisfying:

$$\mathbf{0} = \mathbf{U}_s^{\mathbf{T}}\mathbf{F} = \mathbf{U}_x \cdot \mathbf{F}_x + \mathbf{U}_y \cdot \mathbf{F}_y = \mathbf{ATF}_x + \mathbf{A}\boldsymbol{\phi}\mathbf{TF}_y \tag{8}$$

Then we can denote $\boldsymbol{\psi} = -\mathbf{F}_x\mathbf{F}_y^{-1}$
The equation above is equal to the following:

$$\begin{aligned} &\mathbf{AT}\boldsymbol{\psi} = \mathbf{A}\boldsymbol{\phi}\mathbf{T} \\ \rightarrow &\mathbf{AT}\boldsymbol{\psi}\mathbf{T}^{-1} = \mathbf{A}\boldsymbol{\phi} \\ \rightarrow &\mathbf{U}_s\boldsymbol{\psi} = \mathbf{U}_y \end{aligned} \tag{9}$$

Then if $\mathbf{A}$ is full-order, it will be like this:

$$\mathbf{T}\boldsymbol{\psi}\mathbf{T}^{-1} = \boldsymbol{\phi} \tag{10}$$

All we have to do is to find a $\boldsymbol{\psi}$ that fits the above equation best.

3 Solving

3.1 What we know

Well,to solve the issue,first figure out what we know.
We have the data on every element.Use that, we can esitimate the $R\{E_s\}$. Where E_s is the matrix made by feature vectors of received data.
Note:E_s is an estimation,not true E_s.As a consequence,we use $\widehat{E_s}$ to denote it.

Thus,we got the $\widehat{E_x}$ and $\widehat{E_y}$.
They are the matrixes made by feature vectors of the singal subspace and noise subspace respectively.

3.2 step1

We have data about $\mathbf{X}$,and we can use it to calculate $\hat{\mathbf{U}}$

3.3 step2

With $\mathbf{U}$,we can select two subarrays.After that, we can have our $\mathbf{U}_x$ and $\mathbf{U}_y$

3.4 step3

In practice,we have the estimates for $\mathbf{U}_x$,$\mathbf{U}_y$.
We can denote them as $\widehat{\mathbf{U}_x}$ and $\widehat{\mathbf{U}_y}$
The rule for the best estimate of $\boldsymbol{\psi}$ is that:!!!!It minimizes the difference between $\mathbf{U}_y$ and $\mathbf{U}_s\boldsymbol{\psi}$

$$\begin{aligned}\hat{\boldsymbol{\psi}}_{\mathbf{LS}} &= \arg\min\left\{||\mathbf{U}_y - \mathbf{U}_s\boldsymbol{\psi}||\right\} \\ &= \arg\min\left\{tr\left\{[\mathbf{U}_y - \mathbf{U}_s\boldsymbol{\psi}]^{\mathbf{H}}\left[\mathbf{U}_y - \mathbf{U}_s\boldsymbol{\psi}\right]\right\}\right\}\end{aligned} \tag{11}$$

Note Here:tr means the sum fo the elements in the diagonal.
To be detailed , the result is that :

$$\hat{\boldsymbol{\psi}}_{\mathbf{LS}} = \left[\hat{\mathbf{U}}_x{}^{\mathbf{H}}\hat{\mathbf{U}}_x\right]^{-1}\hat{\mathbf{U}}_x{}^{\mathbf{H}}\hat{\mathbf{U}}_y \tag{12}$$

3.5 step4

Find eigenvalues of $\hat{\boldsymbol{\psi}}_{\mathbf{LS}}$, and we can denote them as $\hat{\lambda_1}, \hat{\lambda_2}, \hat{\lambda_3}, \dots\dots \hat{\lambda_D}$

3.6 step5

Like MUSIC, we can estimate DOA.